Randomness –

A computational complexity view

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Plan of the talk

- Computational complexity
  -- efficient algorithms, hard and easy problems,
  P vs. NP
- The power of randomness
  -- in saving time
- The weakness of randomness
  -- what is randomness ?
  -- the hardness vs. randomness paradigm
- The power of randomness
  -- in saving space
  -- to strengthen proofs

Easy and Hard Problems

asymptotic complexity of functions

Multiplication	Factoring
mult(23,67) = 1541	factor(1541) = (23,67)
grade school algorithm:	best known algorithm:
n^2 steps on n digit inputs	$\exp(\sqrt{n})$ steps on n digits
EASY	HARD?
P – Polynomial time algorithm	-- we don't know! -- the whole world thinks so!

Map Coloring and P vs. NP

Input: planar map M (with n countries)

2-COL: is M 2-colorable? **Easy**

3-COL: is M 3-colorable? **Hard?**

4-COL: is M 4-colorable? **Trivial**

Thm: If 3-COL is Easy then Factoring is Easy

- **Thm [Cook-Levin '71, Karp '72]:** 3-COL is NP-complete
- …. Numerous equally hard problems in all sciences

P vs. NP problem **Formal:** Is 3-COL Easy?

Informal: Can creativity be automated

Fundamental question #1

Is NP≠P ? More generally how fast can we solve:

- **Factoring integers**
- **Map coloring**
- **Satisfiability of Boolean formulae**
- **Computing the Permanent of a matrix**
- **Computing optimal Chess/Go strategies**
- **…….**

Best known algorithms: exponential time/size.

Is exponential time/size necessary for some?

The Power of Randomness

Host of problems for which:

- We have probabilistic polynomial time algorithms

- We (still) have *no* deterministic algorithms of subexponential time.

Coin Flips and Errors

Algorithms will make decisions using coin flips

0111011000010001110101010111…

(flips are independent and unbiased)

When using coin flips, we'll guarantee:

"task will be achieved, with probability >99%"

Why tolerate errors?

- **We tolerate uncertainty in life**
- **Here we can reduce error arbitrarily**

Number Theory: Primes

Problem 1 [Gauss]: Given $x \in [2^n, 2^{n+1}]$, is x prime?

1975 [Solovay-Strassen, Rabin] : Probabilistic

2002 [Agrawal-Kayal-Saxena]: Deterministic !!

Problem 2: Given n, find a prime in $[2^n, 2^{n+1}]$

Algorithm: Pick at random $x_1, x_2, \ldots, x_{1000n}$

For each x apply primality test

Algebra: Polynomial Identities

Is det($\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$) $- \Pi_{i<k} (x_i - x_k) \equiv 0$?

Theorem [Vandermonde]: YES

Given (implicitly, e.g. as a formula) a polynomial p of degree d. Is $p(x_1, x_2, \ldots, x_n) \equiv 0$?

Algorithm [Schwartz-Zippel '80] :

Pick r_i indep at random in {1,2,…,100d}

$p \equiv 0 \;\Rightarrow\;$ Pr[$p(r_1, r_2, \ldots, r_n) = 0$] $=1$

$p \neq 0 \;\Rightarrow\;$ Pr[$p(r_1, r_2, \ldots, r_n) \neq 0$] $> .99$

Applications: Program testing

Analysis: Fourier coefficients

Given (implicitely) a function $f:(Z_2)^n \rightarrow \{-1,1\}$

(e.g. as a formula), and $\varepsilon>0$,

Find all characters χ such that $|<f,\chi>| \geq \varepsilon$

Comment : At most $1/\varepsilon^2$ such χ

Algorithm [Goldreich-Levin '89] :

...adaptive sampling... Pr[success] > .99

[AGS] : Extension to other Abelian groups.

Applications: Coding Theory, Complexity

Geometry: Estimating Volumes

Given (implicitly) a convex body K in R^d (d large!)

(e.g. by a set of linear inequalities)

Estimate volume (K)

Comment: Computing volume(K) exactly is #P-comple

Algorithm [Dyer-Frieze-Kannan '91]:

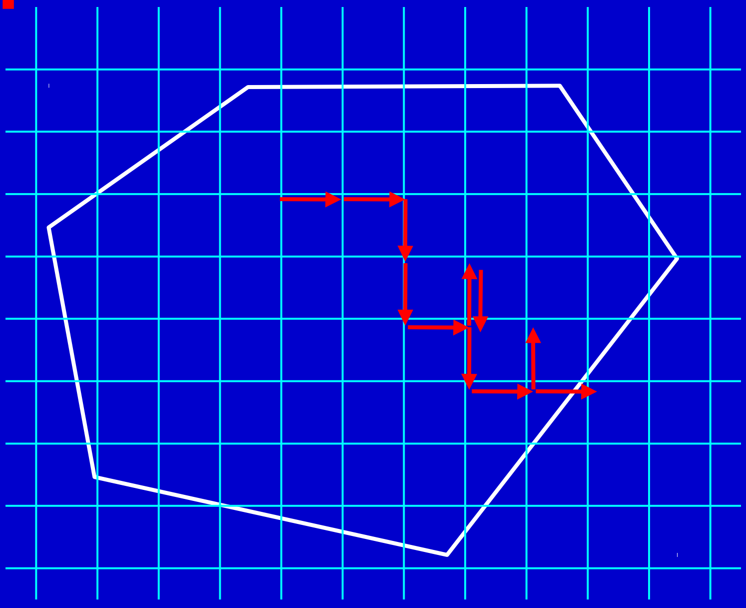
Approx counting ≈ random sampling

Random walk inside K.

Rapidly mixing Markov chain.

Analysis:

Spectral gap ≈ isoperimetric inequality



Fundamental question #2

Does randomness help ?

Are there problems with probabilistic polytime algorithm but no deterministic one?

Conjecture 2: YES

Fundamental question #1

Does NP require exponential time/size ?

Conjecture 1: YES

Theorem: One of these conjectures is false!

Hardness vs. Randomness

Theorems [Blum-Micali,Yao,Nisan-Wigderson,

Impagliazzo-Wigderson…] :

If there are natural hard problems

Then randomness can be efficiently eliminated.

Theorem [Impagliazzo-Wigderson '98]

NP requires exponential *size* circuits ⇒

every probabilistic polynomial-time

algorithm has a deterministic

Computational Pseudo-Randomness

input → **algorithm** → output

n

many unbiased independent

pseudorandom if for *every* **efficient** algorithm, for every input,

output ≈ output

input → **algorithm** → output

n

many biased dependent

efficient deterministic

pseudo-random generator

k ~ c log n

few

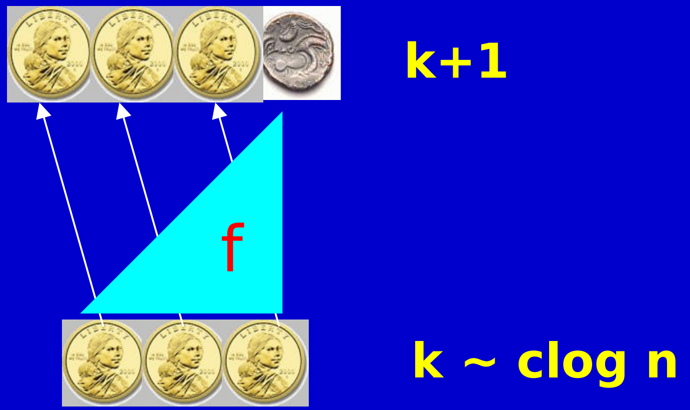
none

Hardness ⇒ Pseudorandomness

Need G: k bits → n bits

NW generator

Show G: k bits → k+1 bits

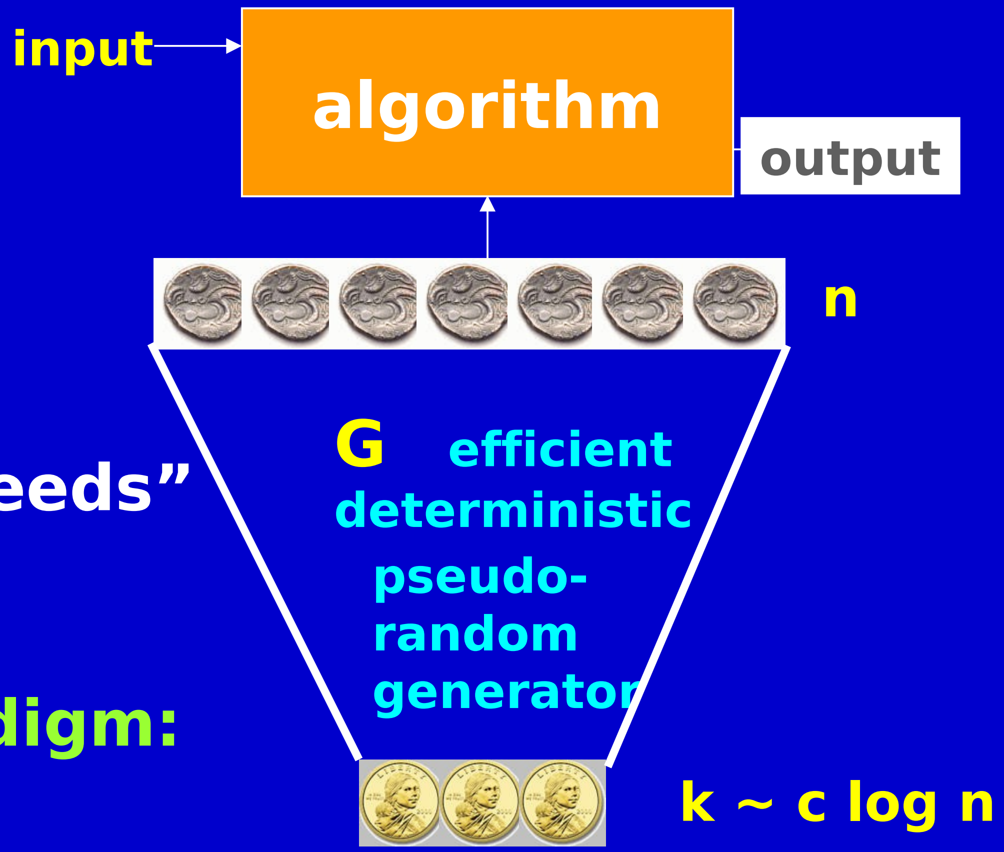


Need: f hard on *random* input Average-case hardnes

Hardness amplification

Have: f hard on *some* input Worst-case hardness

Derandomization



eterministic algorithm:

- Try all possible $2^k = n^c$ "seeds"
- Take majority vote

seudorandomness paradigm:

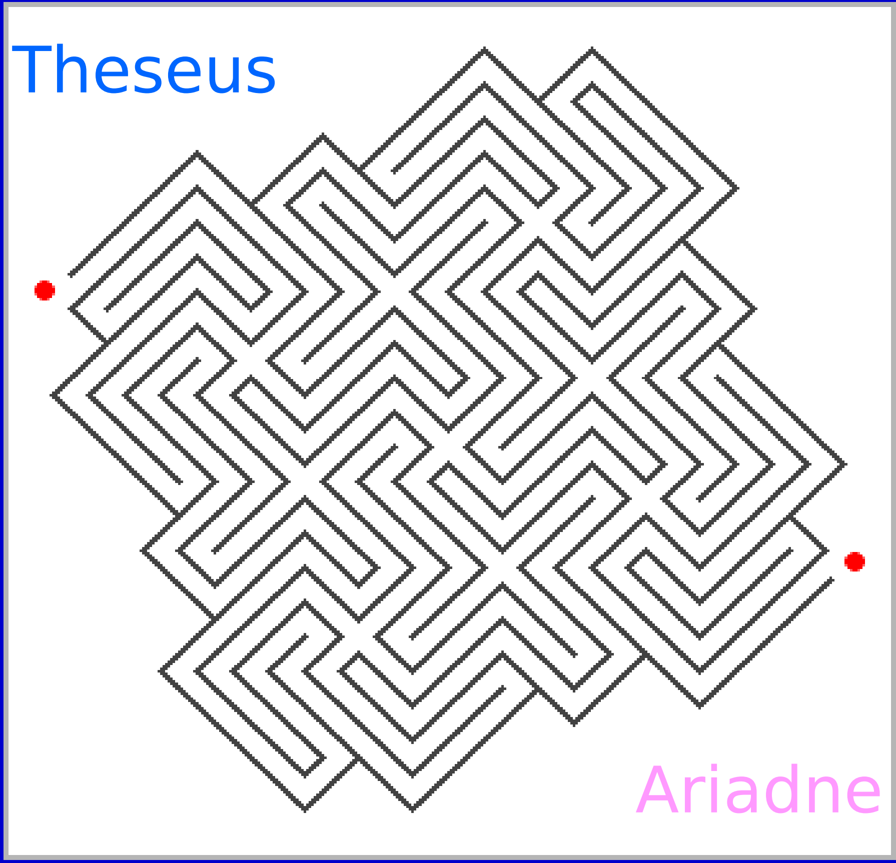
an derandomize specific
lgorithms without assumptions!
.g. Primality Testing & Maze exploration

Randomness and space complexity

Getting out of mazes (when your memory is weak)

n–vertex maze/graph

Only a local view (logspace)



Crete, ~1000 BC

Theorem [Aleliunas-Karp-Lipton-Lovasz-Rackoff '80]: **A random walk will visit every vertex in n^2 steps (with probability >99%)**

Theorem [Reingold '06] : **A deterministic walk, computable in logspace, will visit every vertex.** **Uses ZigZag expanders [Reingold-Vadhan-Wigderson '02]**

The power of pandomness

in Proof Systems

Probabilistic Proof System

[Goldwasser-Micali-Rackoff, Babai ‘85]

Is a mathematical statement claim true? E.g.

claim: “No integers x, y, z, n>2 satisfy $x^n + y^n = z^n$ “

claim: “The Riemann Hypothesis has a 200 page proof”

probabilistic

An efficient Verifier V(claim, argument) satisfies:

always

*) If claim is true then V(claim, argument) = TRUE

for some argument

(in which case claim=theorem, argument=proof)

with probability > 99%

**) If claim is false then V(claim, argument) =

Remarkable properties of Probabilistic Proof Systems

- Probabilistically Checkable Proofs (PCPs)

- Zero-Knowledge (ZK) proofs

Probabilistically Checkable Proofs (PCPs)

claim: The Riemann Hypothesis

Prover: (argument)

Verifier: (editor/referee/amateur)

Verifier's concern: Is the argument correct?

PCPs: Ver reads 100 (random) bits of argument.

Th[Arora-Lund-Motwani-Safra-Sudan-Szegedy'90]

Every proof can be eff. transformed to a PCP

Zero-Knowledge (ZK) proofs

[Goldwasser-Micali-Rackoff ‘85]

claim: The Riemann Hypothesis

Prover: (argument)

Verifier: (editor/referee/amateur)

Prover’s concern: Will Verifier publish first?

ZK proofs: argument reveals only correctness!

Theorem [Goldreich-Micali-Wigderson ‘86]:

Every proof can be efficiently transformed to a ZK proof, *assuming*

Conclusions & Problems

When resources are limited, basic notions get new meanings (randomness, learning, knowledge, proof, …).

- Randomness is in the eye of the beholder.
- Hardness can generate (good enough) randomness.
- Probabilistic algs seem powerful but probably are not.
- Sometimes this can be proven! (Mazes,Primality)
- Randomness is essential in some settings.

Is Factoring HARD? Is electronic commerce secure?

Is Theorem Proving Hard? Is $P \neq NP$? Can creativity