Potential Games

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Cournot Competition

- There is more than one firm and all firms produce a homogeneous product.
- Firms do not cooperate.
- Firms have market power, i.e. each firm's output decision affects the good's price.
- The number of firms is fixed.
- Firms compete in quantities, and choose quantities simultaneously.
- The firms are economically rational and act strategically, usually seeking to maximize profit given their competitors' decisions.

Example 1: Cournot Competition

- n firms: $1, 2, \ldots, n$.
- Firm i chooses a quantity q_i, cost function $c_i(q_i) = cq_i$. Total quality produced: $Q = \sum_{i=1}^{n} q_i$.
- Inverse demand function (price): $F(Q)$, $Q > 0$.
- Profit function for firm i: $\Pi_i(q_1, \ldots, q_2) = F(Q)q_i - cq_i$.
- Define a function P:

$$P(q_1, q_2, \ldots, q_n) = q_1 q_2 \ldots q_n (F(Q) - c).$$

- For all i, for all $q_{-i} \in \mathbb{R}_+^{n-1}$, for all $q_i, x_i \in \mathbb{R}_+$,

$$\Pi(q_i, q_{-i}) - \Pi(x_i, q_{-i}) > 0 \quad \text{iff} \quad P(q_i, q_{-i}) - P(x_i, q_{-i}) > 0.$$

- P is an *ordinal potential function*.

Example 2: Cournot competition

- Cost functions arbitrarily differentiable $c_i(q_i)$.
- Inverse demand function $F(Q) = a - bQ$, $a, b > 0$.
- Define a function P^*:

$$P^*(q_1, \ldots, q_n) = a \sum_{j=1}^{n} q_j - b \sum_{j=1}^{n} q_j^2 - b \sum_{1 \leq i < j \leq n} q_i q_j - \sum_{j=1}^{n} c_j(q_j).$$

- Then, for all i, for all $q_{-i} \in \mathbb{R}_+^{n-1}$, for all $q_i, x_i \in \mathbb{R}_+$,

$$\Pi(q_i, q_{-i}) - (x_i, q_{-i}) = P^*(q_i, q_{-i}) - P^*(x_i, q_{-i}).$$

- P^* is a *potential function*.

Potential Games

- $\Gamma(u^1, u^2, \ldots, u^n)$ a game in strategic form.
- $N = \{1, 2, \ldots, n\}$ the set of players.
- Y^i the set of strategies of player i and $Y = Y^1 \times Y^2 \times \ldots \times Y^n$.
- $u^i : Y \to \mathbb{R}$ the payoff function of player i.

Ordinal Potential

$P : Y \to \mathbb{R}$ *is an* **ordinal potential** *function if,* $\forall\ i \in N$, $\forall\ y^{-i} \in Y^{-i}$,

$$u^i(y^{-i}, x) - u^i(y^{-i}, z) > 0 \quad \textit{iff} \quad P(y^{-i}, x) - P(y^{-i}, z) > 0$$

$\forall x, z \in Y^i$.

- Let $w = (w^i)_{i \in N}$ be a vector of positive numbers (weights).

w-Potential

*$P : Y \to \mathbb{R}$ is a w-**potential** function if, $\forall\ i \in N$, $\forall\ y^{-i} \in Y^{-i}$,*

$$u^i(y^{-i}, x) - u^i(y^{-i}, z) = w^i(P(y^{-i}, x) - P(y^{-i}, z))$$

$\forall x, z \in Y^i$.

- When not interested in particular weights we say that P is a **weighted potential**.

Exact Potential

$P : Y \to \mathbb{R}$ is a **potential** *function if it is a w-potential with $w^i = 1$ for every $i \in N$.*
Alternatively, $\forall\ i \in N$, $\forall\ y^{-i} \in Y^{-i}$,

$$u^i(y^{-i}, x) - u^i(y^{-i}, z) = P(y^{-i}, x) - P(y^{-i}, z)$$

$\forall x, z \in Y^i$.

Example:
The Prisoner's Dilemma game G with

$$G = \begin{pmatrix} (1,1) & (9,0) \\ (0,9) & (6,6) \end{pmatrix}$$

admits a potential

$$P = \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}.$$

- The set of all strategy profiles that maximize the potential P is a subset of the equilibria set.
- The potential function is uniquely defined up to an additive constant (i.e. if P_1, P_2 are potentials for the game Γ, then there is a constant c such that $P_1(y) - P_2(y) = c, \forall y \in Y$).
- Thus, the argmax set of the potential does not depend on a particular potential function.
- The argmax set of P can be used to predict equilibrium points, in some cases.

Corollary

Every finite ordinal potential game possesses a pure-strategy equilibrium.

Finite Improvement Property

Path

*A **path** in Y is a sequence $\gamma = (y_0, y_1, \ldots)$ such that $\forall k \geq 1$ there exists a unique player i such that $y_k = (y_{k-1}^{-i}, x)$ for some $x \neq y_{k-1}^{i}$.*

Improvement Path

*A path γ is an **improvement path** if $\forall k \geq 1$, $u^i(y_k) > u^i(y_{k-1})$, i is the unique player with the above property at step k.*

Finite Improvement Property (FIP)

A game has the FIP if every improvement path is finite.

- Every maximal Finite Improvement Path terminates in an equilibrium point.
- Every finite ordinal potential game has the FIP.
- Having the FIP is not equivalent to having an (ordinal) potential.

Generalized Ordinal Potential

$P : Y \to \mathbb{R}$ *is a* **generalized ordinal potential**, *if* $\forall x, z \in Y^i$,

$$u^i(y^{-i}, x) - u^i(y^{-i}, z) > 0 \implies P(y^{-i}, x) - P(y^{-i}, z) > 0.$$

$\forall x, z \in Y^i$

- A finite game Γ has the FIP $\iff$ Γ has a generalized ordinal potential.

- Finite path $\gamma = (y_0, y_1, \ldots, y_N)$, $v = (v^1, v^2, \ldots, v^n)$. Define:

$$I(\gamma, v) = \sum_{k=1}^{n} [v^{i_k}(y_k) - v^{i_k}(y_{k-1})],$$

where i_k is the unique deviator at step k.
- Closed path:$y_0 = y_N$.
- Simple closed path: $y_l \neq y_k$ for every $0 \leq l \neq k \leq N - 1$ and $y_0 = y_N$.
- Length of simple closed path: The number of distinct vertices in it, $l(\gamma)$.

Theorem

Γ is a game in strategic form. The following are equivalent:

1. *Γ is a potential game.*
2. *$I(\gamma, u) = 0$ for every finite closed path γ.*
3. *$I(\gamma, u) = 0$ for every finite simple closed path γ.*
4. *$I(\gamma, u) = 0$ for every finite simple closed path γ of length 4.*

Proof.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4): obvious.
(1) $\Longrightarrow$ (2): If P is a potential for Γ and $\gamma = (y_0, y_1, \ldots, y_N)$ a closed path, then by the definition of the potential,

$$I(\gamma, u) = I(\gamma, (P, P, \ldots, P)) = P(y_N) - P(y_0) = 0.$$

Proof (cont.)

(2) $\Longrightarrow$ (1): $I(\gamma, u) = 0$ for every closed path γ. Fix a $z \in Y$.

- For every two paths γ_1, γ_2 that connect z to a $y \in Y$, $I(\gamma_1, u) = I(\gamma_2, u)$.
- Indeed, if $\gamma_1 = (z, y_1, \ldots, y_N)$, $\gamma_2 = (z, z_1, \ldots, z_M)$ and $y_N = z_M = y$, then μ is the closed path

  $$\mu = (z, y_1, \ldots, y_N, z_{M-1}, \ldots, z)$$

  and $I(\mu, u) = 0 \Rightarrow I(\gamma_1, u) = I(\gamma_2, u)$.
- For every $y \in Y$, $\gamma(y)$ is the path connecting z to y.
- Define $P(y) = I(\gamma(y), u)$, $\forall y \in Y$.

Proof (cont.)

- P is a potential for Γ.
- $P(y) = I(\gamma, u)$, for every γ that connects z to y.
- $i \in N$, $y^{-i} \in Y^{-i}$, $a \neq b \in Y^i$.
- $\gamma = (z, y_1, \ldots, (y^{-i}, a))$ and $\mu = (z, y_1, \ldots, (y^{-i}, a), (y^{-i}, b))$.
- Then, we have

$$P(y^{-i}, b) - P(y^{-i}, a) = I(\mu, u) - I(\gamma, u) = u^i(y^{-i}, b) - u^i(y^{-i}, a).$$

Proof.

Proof (cont.) (4) $\Longrightarrow$ (2) $I(\gamma, u) = 0$ for every γ with $l(\gamma) = 4$.

- If $I(\gamma, u) \neq 0$ for a closed path γ, then $l(\gamma) = N \geq 5$.
- We can assume that $I(\mu, u) = 0$ whenever $l(\mu) < N$.
- $\gamma = (y_0, y_1, \ldots, y_N)$ and $i(j)$ the unique deviator at step j: $y_{j+1} = (y_j^{-i(j)}, x(i(j)))$.
- Assume $i(0) = 1$. Since $y_N = y_0$, $\exists\ 1 \leq j \leq N-1$: $i(j) = 1$.
- If $i(1) = 1$, let $\mu = (y_0, y_2, \ldots, y_N)$. Then $I(\mu, u) = I(\gamma, u) \neq 0$ but $l(\mu) < N$. Contradiction! The same holds if $i(1) = N-1$.
- Thus, $2 \leq j \leq N-2$.

Proof (cont.)

- $\mu = (y_0, y_1, \ldots, y_{j-1}, z_j, y_{i+1}, \ldots, y - N)$ where

$$z_j = (y_{j-1}^{-[i(j-1),1]}, y_{j-1}^{i(j-1)}, y_{j+1}^{1}).$$

- Then,

$$I((y_{j-1}, y_j, y_{j+1}, z_j), u) = 0.$$

- $I(\mu, u) = I(\gamma, u)$ and $i(j-1) = 1$.
- Continuing recursively we get a contradiction!

□

Congestion Games

- $N = \{1, 2, \ldots, n\}$ the set of players.
- $M = \{1, 2, \ldots, m\}$ the set of facilities.
- Σ^i the set of strategies for player i.
  $A^i \in \Sigma^i$, non-empty set.
  $\Sigma = \times_{i \in N} \Sigma^i$.
- c_j the vector of payoffs, $j \in M$.
  $c_j(k)$ the payoff to each user of facility j if there are exactly k users.
- $\sigma_j(A) = \sharp\{i \in N : j \in A^i\}$, number of users of facility j.

Theorem

Every congestion game is a potential game.

Proof.

For each $A \in \Sigma$ define

$$P(A) = \sum_{j \in \cup_{i=1}^{n} A^i} \left(\sum_{l=1}^{\sigma_j(A)} c_j(l) \right).$$

P is a potential. □

Theorem

Every finite potential game is isomorphic to a congestion game.

thank you!