Price of anarchy in auctions & the smoothness framework

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Algorithmic Game Theory 2016
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Introduction:

The price of anarchy in auctions

COMPLETE INFORMATION GAMES

Example: Chicken game

	stay	swerve
stay	(-10,-10)	(1,-1)
swerve	(-1,1)	(0,0)

The strategy profile (stay, swerve) is a mutual best response, a Nash equilibrium.

A **Nash equilibrium** in a game of *complete* information is a strategy profile where each player's strategy is a best response to the strategies of the other players as given by the strategy profile

- **pure strategies** : correspond directly to actions in the game
- **mixed strategies** : are randomizations over actions in the game

INCOMPLETE INFORMATION GAMES (AUCTIONS)

- Each agent has some private information (agent's **valuation** v_i) and this information affects the payoff of this agent in the game.
- **strategy** in a incomplete information auction = a function $b_i(\cdot)$ that maps an agent's type to any **bid** of the agent's possible bidding actions in the game

strategy

$$v_i \xRightarrow{b_i(\cdot)} b_i(v_i)$$

valuation **bid**

Example: Second Price Auction

A *strategy* of player i maps valuation to bid $\mathrm{b_i(v_i)} = \text{"bid } \mathrm{v_i} \text{"}$

*This strategy is also truthful.

FIRST PRICE AUCTION OF A SINGLE ITEM

- a single item to sell
- n players - each player i has a private **valuation** $v_i \sim F_{\boldsymbol{i}}$ for the item.
- distribution $\boldsymbol{F}$ is known and valuations v_i are drawn independently

First Price Auction

1. the auction winner is the maximum bidder
2. the winner pays his bid

F is the product distribution
$$\boldsymbol{F} \equiv F_1 \times \cdots \times F_n$$

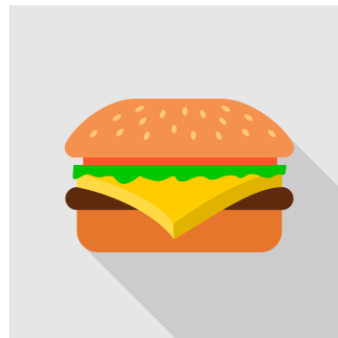
Then, $\boldsymbol{F}_{-i} | v_i = \boldsymbol{F}_{-\mathrm{i}}$

Player's goal: maximize **utility** = valuation−price paid

FIRST PRICE AUCTION: Symmetric

Two bidders, independent valuations with uniform distribution U([0,1])

value v_1, bid b_1



value v_2, bid b_2

If the cat bids **half her value**, *how should you bid?*

Your expected utility: $\mathbf{E}[u_1] = (v_1 - b_1) \cdot \mathbf{P}[you\ win]$

$$\mathbf{P}[you\ win] = \mathbf{P}\ [b_2 \leq b_1] = 2b_1 \Rightarrow \mathbf{E}[u_1] = 2v_1 b_1 - 2b_1^2$$

Optimal bid: $\frac{d}{db_1}\mathbf{E}[u_1] = 0 \ \Rightarrow b_1 = \frac{v_1}{2}$ **BNE**

BAYES-NASH EQUILIBRIUM (**BNE**) + PRICE OF ANARCHY (**PoA**)

A **Bayes-Nash equilibrium (BNE)** is a strategy profile where if for all i $b_i(v_i)$ is a best response when other agents play $b_{-i}(v_{-i})$ with $v_{-i} \sim \mathbf{F_{-i}}|v_i$ (conditioned on v_i)

Price of Anarchy (PoA) = the worst-case ratio between the objective function value of an equilibrium and of an optimal outcome

Example of an auction objective function:
Social welfare = the valuation of the winner

FIRST PRICE AUCTION: Symmetric vs Non-Symmetric

Symmetric Distributions [two bidders $U([0,1])$]

- $b_1(v_1) = \frac{v_1}{2}$, $b_2(v_2) = \frac{v_2}{2}$ is BNE
- the player with the highest valuation wins in BNE $\Rightarrow$
  first-price auction **maximizes social welfare**

Non-Symmetric Distributions [two bidders $v_1 \sim U([0,1]),\ v_2 \sim U([0,2])$]

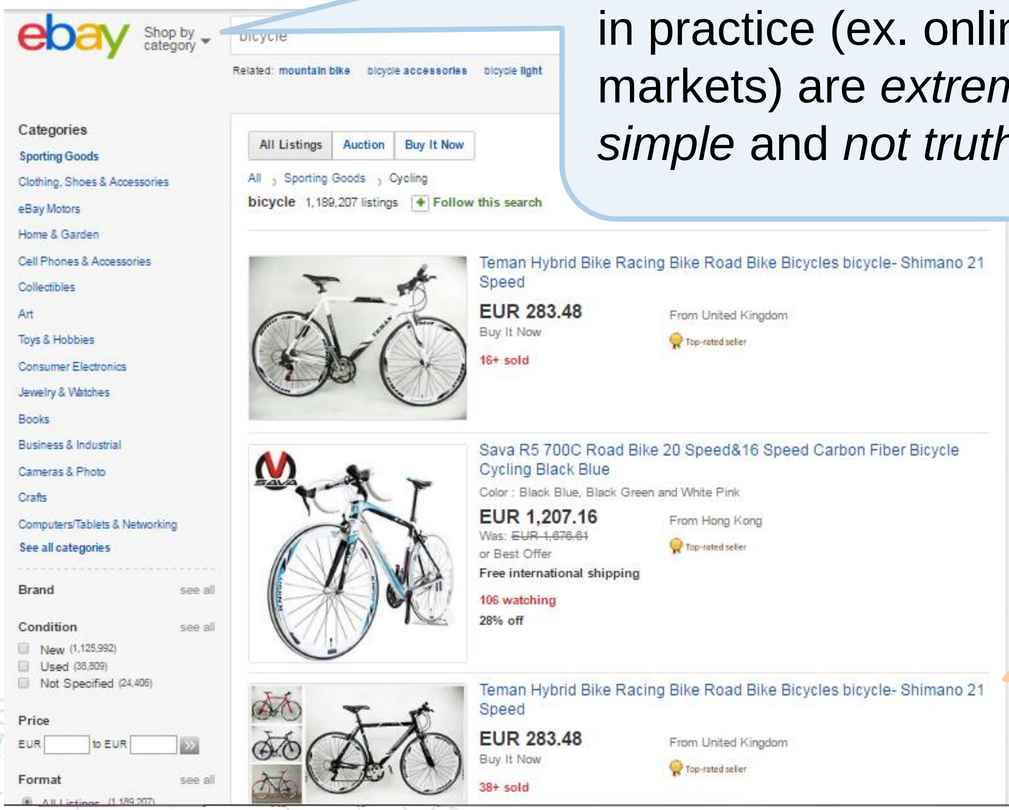
- $b_1(v_1) = \frac{2}{3v_1}\left(2 - \sqrt{4 - 3v_1^2}\right)$, $b_2(v_2) = \frac{2}{3v_2}\left(-2 + \sqrt{4 + 3v_2^2}\right)$ is BNE
- player 1 may win in cases where $v_2 > v_1 \Rightarrow$ **PoA>1**

The smoothness framework

MOTIVATION: Simple and... not-so-simple auctions

Simple! Single item second price auction

Simple?



Typical mechanisms used in practice (ex. online markets) are *extremely simple* and *not truthful*!

How *realistic* is the assumption that mechanisms run *in isolation*, as traditional mechanism design has considered?

COMPOSITION OF MECHANISMS

Simultaneous Composition of m Mechanisms
The player reports a bid at each mechanism M_j

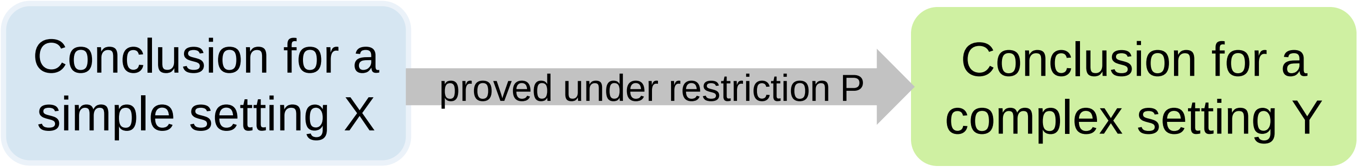
Sequential Composition of m Mechanisms
The player can base the bid he submits at mechanism M_j on the *history* of the submitted bids in previous mechanisms.

<u>Reducing analysis of complex setting to simple setting.</u>

How to design mechanisms so that the efficiency guarantees for a **single** mechanism (when studied in isolation) carry over to the same or approximately the same guarantees for a market **composed** of such mechanisms?

<u>Key question</u>

What properties of **local** *mechanisms guarantee* **global** *efficiency in a market composed of such mechanisms?*



SMOOTHNESS

Smooth auctions

An auction game is $(\boldsymbol{\lambda}, \boldsymbol{\mu})$**-smooth** if ∃ a bidding strategy $\mathbf{b}^*$ s.t. $\forall \mathbf{b}$

$$\sum_i u_i\left(b_i^*, b_{-i}\right) \geq \boldsymbol{\lambda} \cdot OPT - \boldsymbol{\mu} \sum_i p_i(\mathbf{b})$$

Smoothness is property of auction not equilibrium!

SMOOTHNESS IMPLIES PoA [PNE]

$$PoA = \frac{OPT(\mathbf{v})}{SW(\mathbf{b})}$$

(λ,μ)-smoothness $\Rightarrow PoA \le \frac{\max(1,\mu)}{\lambda}$

THEOREM

The (λ,μ)-smoothness property of an auction implies that a Nash equilibrium strategy profile $\mathbf{b}$ satisfies $\max\{1,\mu\}\, SW(\mathbf{b}) \ge \lambda \cdot OPT$

Proof. Let $\mathbf{b}$: a Nash strategy profile,

$\mathbf{b}^*$: a strategy profile that satisfies smoothness

$\mathbf{b}$ Nash strategy profile $\Rightarrow u_i(\mathbf{b}) \ \ge \ u_i(b_i^*, b_{-i})$

Summing over all players: $\sum_i u_i(\mathbf{b}) \ge \sum_i u_i(b_i^*, b_{-i})$

By auction smoothness: $\sum_i u_i(\mathbf{b}) \ge \boldsymbol{\lambda} \cdot OPT - \boldsymbol{\mu} \sum_i p_i(\mathbf{b})$

$\Rightarrow \sum_i u_i(\mathbf{b}) + \boldsymbol{\mu} \sum_i p_i(\mathbf{b}) \ge \boldsymbol{\lambda} \cdot OPT \Rightarrow \max\{1,\mu\}\, SW(\mathbf{b}) \ge \lambda \cdot OPT$

A vector of strategies **s** is said to be a **Nash equilibrium** if for each player *i* and each strategy s_i':

$$u_i(\mathbf{s}) \ge u_i(s_i', s_{-i})$$

An auction game is **(λ,μ)-smooth** if ∃ a bidding strategy $\mathbf{b}^*$ s.t. ∀$\mathbf{b}$

$$\sum_i u_i(b_i^*, b_{-i}) \ge \lambda \cdot OPT - \mu \sum_i p_i(\mathbf{b})$$

SMOOTHNESS IMPLIES PoA [**BNE!**]

$$PoA = \frac{E[OPT(\mathbf{v})]}{E[SW(\mathbf{b}(\mathbf{v}))]}$$

(λ,μ)-smoothness $\Rightarrow$ ***BNE PoA*** $\leq \frac{\max(1,\mu)}{\lambda}$

THEOREM : Generalization to Bayesian settings

The (λ, μ)-smoothness property of an auction (with an $\mathbf{b}^*$ such that b_i^* depends only on the value of player i) implies that a Bayes-Nash equilibrium strategy profile $\mathbf{b}$ satisfies $\max\{1, \mu\}\ \mathbf{E}[SW(\mathbf{b})] \geq \lambda \cdot \mathbf{E}[OPT]$

A vector of strategies **s** is said to be a **Bayes-Nash equilibrium** if for each player *i* and each strategy s_i', maximizes utility (conditional on valuation v_i)

$$\mathrm{E}_v\left[u_i(\mathbf{s})|v_i\right] \geq \mathrm{E}_v\left[u_i(s_i', s_{-i})|v_i\right]$$

Complete information PNE to BNE with correlated values: Extension Theorem 1

Conclusion for a simple setting X

POA extension theorem

Conclusion for a complex setting Y

Complete information
Pure Nash Equilibrium
$\mathbf{v} = (v_1, \ldots, v_n)$: common knowledge

$$PoA = \frac{OPT(\mathbf{v})}{SW(\mathbf{b})}$$

Incomplete information
Bayes-Nash Equilibrium
with asymmetric correlated valuations

$$PoA = \frac{E[OPT(\mathbf{v})]}{E[SW(\mathbf{b}(\mathbf{v}))]}$$

FPA AND SMOOTHNESS

An auction game is (λ, μ)-smooth if $\exists$ a bidding strategy $\mathbf{b}^*$ s.t. $\forall \mathbf{b}$

$$\sum_i u_i\left(b_i^*, b_{-i}\right) \geq \lambda \cdot OPT - \mu \sum_i p_i(\mathbf{b})$$

LEMMA

First Price Auction (complete information) of a single item is $(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{1})$-smooth

Proof. We'll prove that $\sum_i u_i\left(b_i^*, b_{-i}\right) \geq \frac{1}{2}\, OPT - \sum_i p_i(\mathbf{b})$.

Let's try the bidding strategy $b_i^* = \frac{v_i}{2}$.

Maximum valuation bidder: $j = \arg\max_i v_i$

- If j wins, $u_j = v_j - b_j^*(v_j) = \frac{v_j}{2} \geq \frac{1}{2} v_j - \sum_i p_i(\mathbf{b}$
- If j loses, $u_j = 0$, and $\sum_i p_i(\mathbf{b}) = \max_i b_i > \frac{1}{2}$

$$\Rightarrow u_j = 0 > \frac{1}{2} v_j - \sum_i p_i(\mathbf{b}).$$

For all other bidders $i \neq j : u_i(b_i^*, b_{-i}) \geq 0$.

Summing up over all players we get

$$\sum_i u_i(b_i^*, b_{-i}) \geq \frac{1}{2} v_j - \sum_i p_i(\mathbf{b}) = \frac{1}{2} OPT - \sum_i p_i(\mathbf{b})$$

First Price Auction of a single item is $(\mathbf{1} - \frac{\mathbf{1}}{\boldsymbol{e}}, \mathbf{1})$-smooth

COMPLETE INFORMATION FIRST PRICE AUCTION : PNE & Complete Information

LEMMA
Complete Information First Price Auction of a single item has PoA ≤ 2

$$PoA = \frac{OPT(\mathbf{v})}{SW(\mathbf{b})}$$

Proof.
Each bidder i can deviate to $b_i = \frac{v_i}{2}$.
We prove that $SW(\mathbf{b}) \geq \frac{1}{2} OPT(\mathbf{v})$.

Complete Information First Price Auction of a single item has **PoA =1.**
But…

First Extension Theorem

FPA (complete info) is $(\mathbf{1} - \frac{\mathbf{1}}{\boldsymbol{e}}, \mathbf{1})$-smooth

- Prove smoothness property of simple setting
- Prove PoA of simple setting via own-based deviations

  FPA (complete info) has PoA ≤ 2
- Use Extension Theorem to prove PoA of target setting

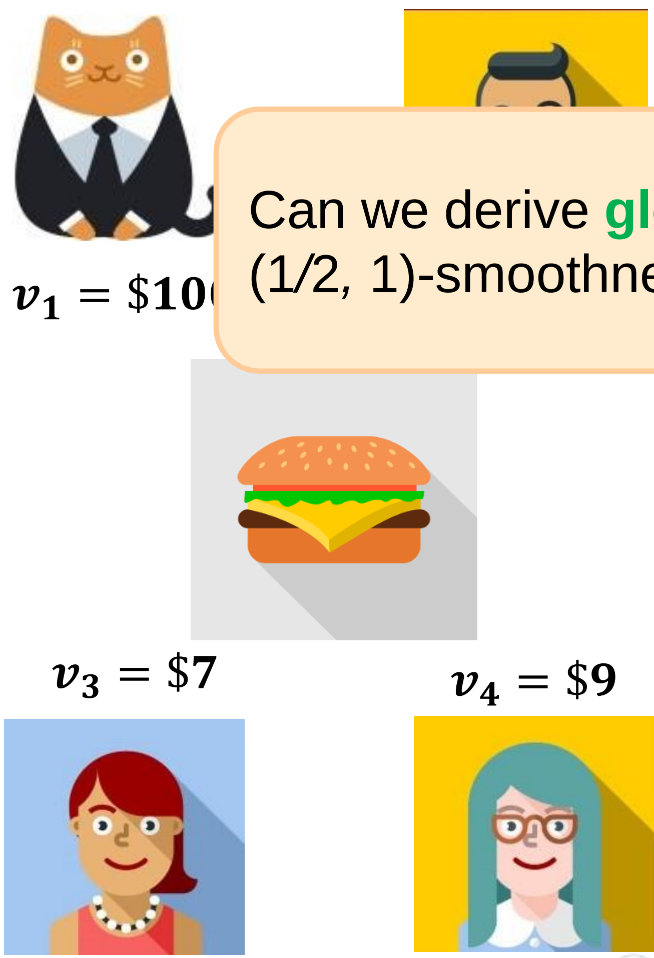
$$\boldsymbol{PoA} \leq \frac{\boldsymbol{e}}{\boldsymbol{e}-\mathbf{1}} \approx \mathbf{1.58}$$

EXTENSION THEOREM 1

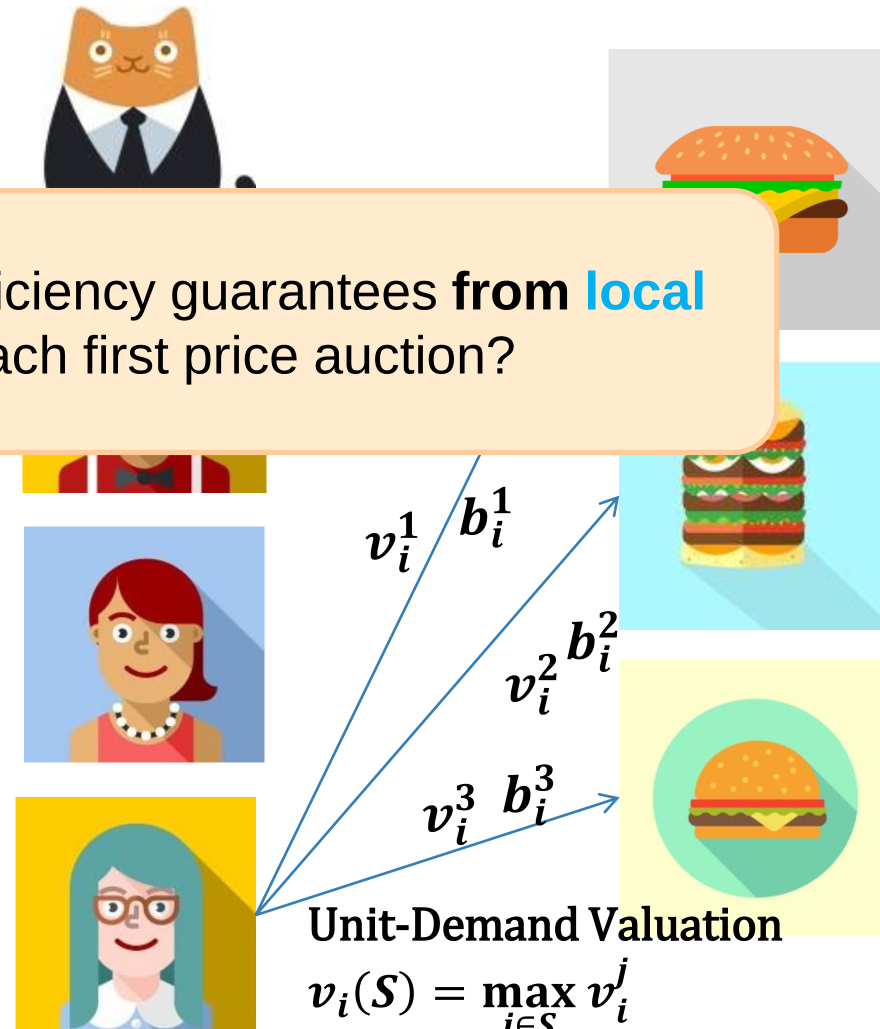
PNE PoA proved by showing (λ, μ) –smoothness property via own-value deviations ⇒ PoA bound of BNE with correlated values $\frac{\max\{\mu,1\}}{\lambda}$

The Composition Framework:
Extension Theorem 2

Simple setting. Single-item first price auction (complete information **PNE**).



Target setting. Simultaneous single-item first price auctions with unit-demand bidders (complete information **PNE**).

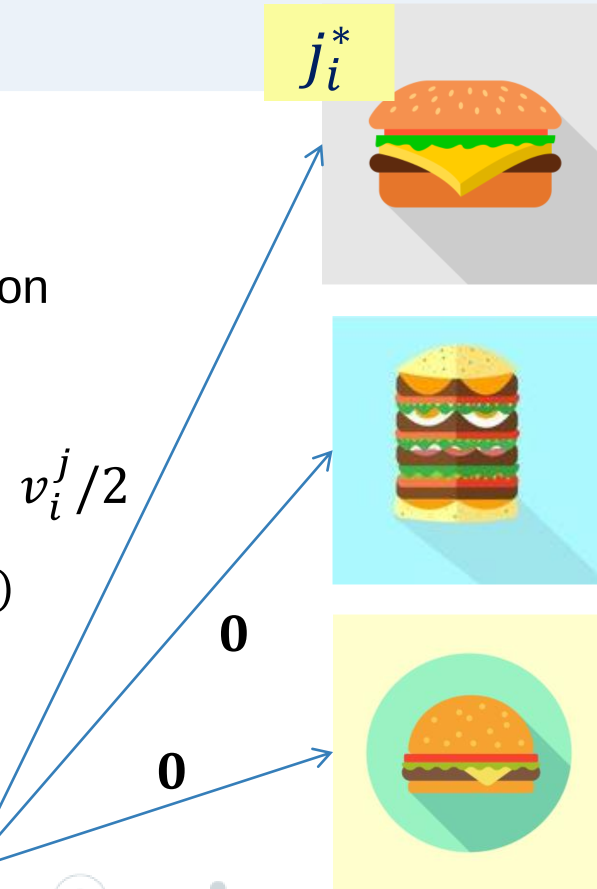


Can we derive **global** efficiency guarantees **from local** (1/2, 1)-smoothness of each first price auction?

FROM SIMPLE LOCAL SETTING TO TARGET GLOBAL SETTING

EXTENSION THEOREM 2

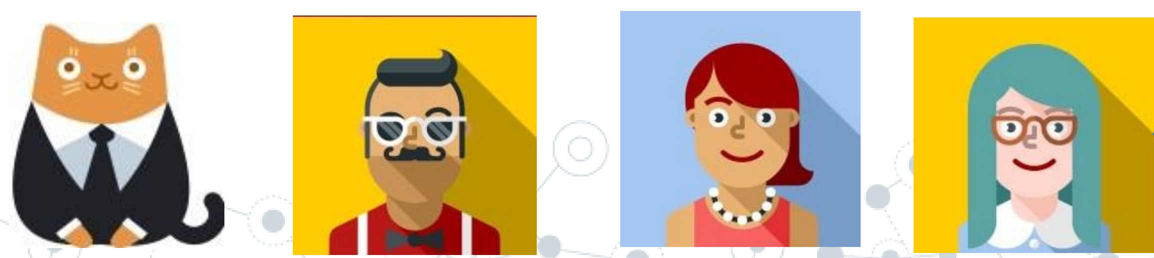
PNE PoA bound of 1-item auction ⇒**PNE** PoA bound of *simultaneous* auctions based on proving **smoothness**



Proof sketch.

Prove **smoothness** of the global mechanism!

- ✓ *Global deviation:* Pick your item in the optimal allocation and perform the smoothness deviation for your local value v_i^j, i.e. $b_i^* = v_i^j/2$.
- ✓ *Smoothness locally:* $u_i(b_i^*, b_{-i}) \geq \frac{v_i^j}{2} - p_{j_i^*}$
- ✓ *Sum over players:* $\sum_i u_i(b_i^*, \mathbf{b_{-i}}) \geq \frac{1}{2} \cdot OPT(\mathbf{v}) - REV(\mathbf{b})$
- ✓ **(1/2, 1)-smoothness property globally**



The Composition Framework:

Extension Theorem 3

FROM SIMPLE LOCAL SETTING TO TARGET GLOBAL SETTING

EXTENSION THEOREM 3

If **PNE** PoA of single-item auction proved via (λ, μ)-smoothness via valuation profile dependent deviation,

$\Rightarrow$ then BNE PoA bound of simultaneous auctions with *submodular* and *independent* valuations also $\max\{\mu, 1\}/\lambda$

Let f be a set function.
f is **submodular** iff
$f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$

BNE PoA of simultaneous first price auctions with submodular and independent bidders $\leq \frac{e}{e-1}$

SUMMARY

Conclusion for a <u>simple</u> setting X → **proved under restrictions** → Conclusion for a <u>complex</u> setting Y

- *X: complete information PNE ⇒ Y: incomplete information BNE*
- *X: single auction ⇒ Y: composition of auctions*

Smooth auctions

An auction game is **(λ, μ)-smooth** if ∃ a bidding strategy $\mathbf{b}^*$ s.t. $\forall \mathbf{b}$

$$\sum_i u_i\left(b_i^*, b_{-i}\right) \geq \lambda \cdot OPT - \mu \sum_i p_i(\mathbf{b})$$

- Applies to "any" auction, not only first price auction.
- Also true for **sequential** auctions.

The Composition Framework

We can combine these two theorems to prove efficiency guarantees when mechanisms are run *in a sequence of rounds and at each round several mechanisms are run simultaneously.*

Simultaneous Composition of m Mechanisms

Suppose that

- each mechanism M_j is (λ, μ) -smooth

- the valuation of each player across mechanisms is XOS.

Then the global mechanism is (λ, μ) -smooth.

Sequential Composition of m Mechanisms

Suppose that

- each mechanism M_j is (λ, μ) -smooth

- the valuation of each player comes from his best mechanism's outcome $v_i(x_i) = \max_j v_{ij}(x_{ij})$.

Then the global mechanism is $(\lambda, \mu + 1)$ −smooth, independent of the information released to players during the sequential rounds.

Applications

APPLICATIONS

Effective Welfare

$$EW(x) = \sum_i \min\{v_i(x_i), B_i\}$$

- m **simultaneous first price** auctions and bidders have **budgets** and fractionally subadditive valuations ⇒ BNE achieves at least $\frac{e-1}{e} \approx 0.63$ of the expected optimal effective welfare
- **Generalized First-Price Auction**: n bidders, m slots. We allocate slots by bid and charge bid per-click. Bidder's utility:

$$u_i(\mathbf{b}) = a_{\sigma(i)}(v_i - b_i)$$

  BNE $PoA \leq 2$
- **Public Goods Auctions**: n bidders, m public projects. Choose a single public project to implement . Each player i has a value v_{ij} if project j is implemented

MECHANISM 3: First price public project auction.

1 Solicit bids b_{ij} from each player i for each project j;
2 For a project $j \in [m]$, let $B_j = \sum_{i \in [n]} b_{ij}$;
3 Pick project $j(b) = \arg\max_{j \in [m]} B_j$;
4 Charge each player his bid for the chosen project $b_{ij(b)}$

APPLICATIONS

- m simultaneous with budgets/sequential **bandwidth allocation** mechanisms
- Second Price Auction ***weakly smooth mechanism*** (λ, $\mu 1$, $\mu 2$) + willingness-to-pay
- **All-pay** auction - proof similar to FPA

MECHANISM 4: Proportional bandwidth allocation mechanism.

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1. Solicit a single bid b_i from each player i;
2. Allocate to player i bandwidth $x_i(b) = \frac{b_i C}{\sum_{j \in N} b_j}$;
3. Charge each player his bid b_i

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