$\exists \mathcal{F} : \mathsf{NP}^{\mathcal{F}} \not\subseteq \mathsf{BQP}^{\mathcal{F}}$

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Presentation Structure

Introduction

Introduction

$$\underbrace{\square\ \square\ \square\ \square\ \square\ \square\ \square\ \square\ \square\ \square\ \square\ \square\ \square\ \square\ \square\ \square}_{16}$$

$$\forall \square : \square \in \{0, 1\}$$

Figure : Our 16-bit computer, with 2^{16} configurations.

Introduction

$$b_1 \; b_2 \; b_3 \; b_4 \; b_5 \; b_6 \; b_7 \; b_8 \; b_9 \; b_{10} \; b_{11} \; b_{12} \; b_{13} \; b_{14} \; b_{15} \; b_{16}$$

$$\forall i \in \{1, 2, \ldots, 16\} : b_i \in \{0, 1\}$$

Figure : Communicating a configuration of a **deterministic** computer.

Introduction

$$\underbrace{b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7\ b_8\ b_9\ b_{10}\ b_{11}\ b_{12}\ b_{13}\ b_{14}\ b_{15}\ b_{16}}_{\text{a 16-bit string} \Leftrightarrow k \in \mathbb{N} \Leftrightarrow p_k = 1}$$

$$\forall i \in \{1, 2, \ldots, 16\} : b_i \in \{0, 1\}$$

Figure : Communicating a configuration of a **deterministic** computer.

Introduction

$$p_1 \; p_2 \; \dots \; p_{2^{16}}$$

$$\forall i \in \left\{1, 2, \dots, 2^{16}\right\} : p_i \in \{0, 1\}$$

$$\sum_{i=1}^{2^{16}} p_i = 1 \Leftrightarrow \exists! k : p_k = 1$$

Figure : Communicating a configuration of a **deterministic** computer.

Introduction

$$p_1 \;\; p_2 \;\; \ldots \;\; p_{2^{16}}$$

$$\forall i \in \left\{1, 2, \ldots, 2^{16}\right\} : p_i \in [0, 1]$$

$$\sum_{i=1}^{2^{16}} p_i = 1$$

Figure : Communicating a configuration of a **probabilistic** computer.

Introduction

$c_1 \ c_2 \ \ldots \ c_{2^{16}}$

$$\forall i \in \left\{1, 2, \ldots, 2^{16}\right\} : c_i \in \mathbb{C}$$

$$\sum_{i=1}^{2^{16}} |c_i|^2 = 1 \qquad (*)$$

Figure : Communicating a configuration of a **quantum** computer.

Quantum Computing 101

Quantum States

Quantum Computing 101: Quantum States

$$\mathcal{B} = \{\mathbf{v}_i\}_{i=1}^{2^{16}}$$

$$\begin{aligned}
\mathbf{v}_1 &= 0\ldots000 \\
\mathbf{v}_2 &= 0\ldots001 \\
&\vdots \\
\mathbf{v}_{2^{16}} &= \underbrace{1\ldots111}_{16}
\end{aligned}$$

Quantum Computing 101: Quantum States

$$\mathbf{v}_1 = \underbrace{0\ldots000}_{16} = \left.\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}\right\} 2^{16}$$

$$\mathbf{v}_2 = \underbrace{0\ldots001}_{16} = \left.\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}\right\} 2^{16}$$

Quantum Computing 101: Quantum States

. . .

Quantum Computing 101: Quantum States

$$\mathbf{v}_{2^{16}-1} = \underbrace{1\ldots110}_{16} = \left.\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}\right\} 2^{16}$$

$$\mathbf{v}_{2^{16}} = \underbrace{1\ldots111}_{16} = \left.\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\right\} 2^{16}$$

Quantum Computing 101: Quantum States

$$\mathcal{B} = \{\mathbf{v}_i\}_{i=1}^{2^{16}}$$

$$\begin{aligned}\mathcal{H}' &= \text{span}\left(\mathcal{B}, \mathbb{C}\right) = \left\{ \sum_{i=1}^{2^{16}} c_i \cdot \mathbf{v}_i \mid \forall i : c_i \in \mathbb{C} \text{ and } \mathbf{v}_i \in \mathcal{B} \right\} \\ &= \mathbb{C}^{2^{16}}\end{aligned}$$

$$\mathcal{H} = \left\{\mathbf{q} \in \mathcal{H}' \mid \|\mathbf{q}\|_2 = 1\right\} \subseteq \mathcal{H}'$$

Quantum Computing 101: Quantum States

$$\mathcal{H} = \left\{ \mathbf{q} \in \mathbb{C}^{2^{16}} \mid \|\mathbf{q}\|_2 = 1 \right\}$$
$$= \text{Our world.}$$
$$\subseteq \mathbb{C}^{2^{16}}$$

Quantum Computing 101: Quantum States

$n \in \mathbb{N}$

$$\mathcal{H} = \left\{ \mathbf{q} \in \mathbb{C}^{2^n} \mid \|\mathbf{q}\|_2 = 1 \right\}$$

Quantum Computing 101: Quantum States

$n \in \mathbb{N}$

$n =$ The number of qubits of our quantum system.

$$\mathcal{H} = \left\{ \mathbf{q} \in \mathbb{C}^{2^n} \mid \|\mathbf{q}\|_2 = 1 \right\}$$

Quantum Computing 101: Quantum States

$$n \in \mathbb{N}$$

$$\mathcal{H} = \left\{ \mathbf{q} \in \mathbb{C}^{2^n} \mid \|\mathbf{q}\|_2 = 1 \right\}$$

Quantum Computing 101: Quantum States

$n \in \mathbb{N}$

$$\mathcal{H} = \left\{ \mathbf{q} \in \mathbb{C}^{2^n} \mid \|\mathbf{q}\|_2 = 1 \right\}$$

$$\mathbf{q} = \left(\sum_{i=1}^{2^{16}} c_i \cdot \mathbf{v}_i \right) \in \mathcal{H} \Leftrightarrow \|\mathbf{q}\|_2 = 1 \Leftrightarrow \sum_{i=1}^{2^n} |c_i|^2 = 1 \qquad (*)$$

Quantum Computing 101: Quantum States

$$|\psi\rangle$$

Quantum Computing 101: Quantum States

$$|\psi\rangle \in \mathcal{H} \subseteq \mathbb{C}^{2^n}$$

Quantum Computing 101: Quantum States

$$
\begin{aligned}
|\psi\rangle &= \text{a ket} \\
&= \text{a column vector}
\end{aligned}
$$

$$
\begin{aligned}
\langle\psi| &= \text{a bra} \\
&= \text{the dual of the ket } |\psi\rangle \\
&= |\psi\rangle^{\dagger} \\
&= \left(|\psi\rangle^{*}\right)^{T} = \left(|\psi\rangle^{T}\right)^{*} \\
&= \text{a row vector}
\end{aligned}
$$

Quantum Computing 101: Quantum States

$$\mathcal{B} = \{\mathbf{v}_i\}_{i=1}^{2^{16}}$$

$$\begin{aligned}
\mathbf{v}_1 &= 0\ldots000 \\
\mathbf{v}_2 &= 0\ldots001 \\
&\vdots \\
\mathbf{v}_{2^{16}} &= 1\ldots111
\end{aligned}$$

Quantum Computing 101: Quantum States

$$\mathcal{B} = \{|v_i\rangle\}_{i=1}^{2^{16}}$$

$$\begin{aligned} |v_1\rangle &= |0\ldots000\rangle \\ |v_2\rangle &= |0\ldots001\rangle \\ &\vdots \\ |v_{2^{16}}\rangle &= |1\ldots111\rangle \end{aligned}$$

Quantum Computing 101: Quantum States

$$\mathcal{B} = \{|v_i\rangle\}_{i=1}^{2^{16}}$$

$$\begin{aligned}
|v_1\rangle &= |0\ldots000\rangle = |1\rangle \\
|v_2\rangle &= |0\ldots001\rangle = |2\rangle \\
&\vdots \qquad\qquad\quad \vdots \\
|v_{2^{16}}\rangle &= |1\ldots111\rangle = \left|2^{16}\right\rangle
\end{aligned}$$

Quantum Computing 101: Quantum States

Example: The qubit.

$$\mathcal{B} = \{|v_i\rangle\}_{i=1}^{2}$$

$$|v_1\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|v_2\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Quantum Computing 101: Quantum States

$$\begin{aligned}\mathcal{I}\left(|\psi\rangle, |\phi\rangle\right) &= \text{inner product}\\ &= \langle\psi|\phi\rangle \in \mathbb{C}\end{aligned}$$

$$\begin{aligned}\mathcal{O}\left(|\psi\rangle, |\phi\rangle\right) &= \text{outer product}\\ &= |\psi\rangle\langle\phi| \in \mathbb{C}^{2^n \times 2^n}\end{aligned}$$

Unitary Evolution

Quantum Computing 101: Unitary Evolution

$$U\,|q_{\text{old}}\rangle = |q_{\text{new}}\rangle$$

$$U^{-1} = U^{\dagger} = (U^{*})^{T} = \left(U^{T}\right)^{*}$$

Quantum Computing 101: Unitary Evolution

$$|q_{\text{initial}}\rangle \in \mathcal{H}$$

Quantum Computing 101: Unitary Evolution

$$U_1 \left|q_{\text{initial}}\right\rangle \in \mathcal{H}$$

Quantum Computing 101: Unitary Evolution

$$U_2 U_1 \left| q_{\text{initial}} \right\rangle \in \mathcal{H}$$

Quantum Computing 101: Unitary Evolution

$$m \in \mathbb{N}$$

$$U_m \cdots U_2 U_1 \left| q_{\text{initial}} \right\rangle \in \mathcal{H}$$

Quantum Computing 101: Unitary Evolution

$$m \in \mathbb{N}$$

$$|q_{\text{final}}\rangle = U_m \cdots U_2 U_1 |q_{\text{initial}}\rangle \in \mathcal{H}$$

Quantum Computing 101: Unitary Evolution

$$m \in \mathbb{N}$$

$$|q_{\text{final}}\rangle = U_m \cdots U_2 U_1 |q_{\text{initial}}\rangle \in \mathcal{H}$$

Figure : Our first quantum algorithm.

Measurements

Quantum Computing 101: Measurements

$$|\psi\rangle = \left(\sum_{i=1}^{2^n} c_i \cdot |v_i\rangle \right) \in \mathcal{H}$$

$n \in \mathbb{N}$

$\forall i : c_i \in \mathbb{C}$ and $|v_i\rangle \in \mathcal{B}$

$$\sum_{i=1}^{2^n} |c_i|^2 = 1 \qquad (*)$$

Quantum Computing 101: Measurements

$$|\psi\rangle = \sum_{i=1}^{2^n} c_i \cdot |v_i\rangle \overset{\text{Measurement}}{\Rightarrow} \exists j : |\psi'\rangle = |v_j\rangle$$

$$\Pr\left[\text{The outcome is } j.\right] = |c_j|^2$$

Quantum Computing 101: Measurements

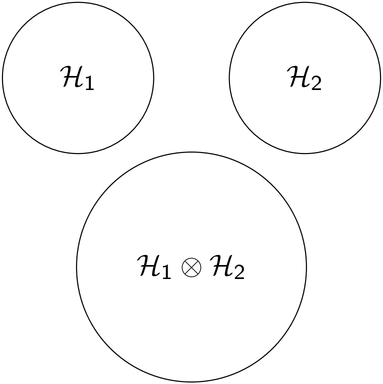
$$|\psi\rangle = \sum_{i=1}^{2^n} c_i \cdot |v_i\rangle \overset{\text{Measurement}}{\Rightarrow} \exists j : |\psi'\rangle = |v_j\rangle$$

$$\overset{\text{Measurement}}{\Rightarrow} |\psi''\rangle = |v_j\rangle$$

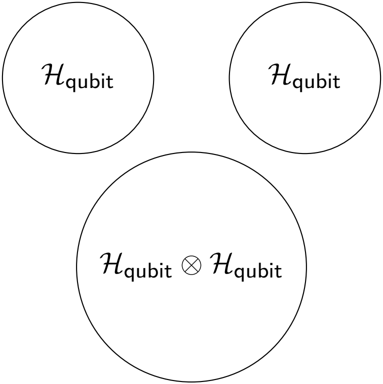
$$\begin{aligned} \Pr[\text{The outcome is } j.] &= |1|^2 \\ &= 1 \end{aligned}$$

Composition

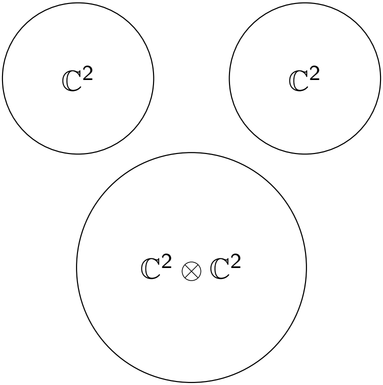
Quantum Computing 101: Composition



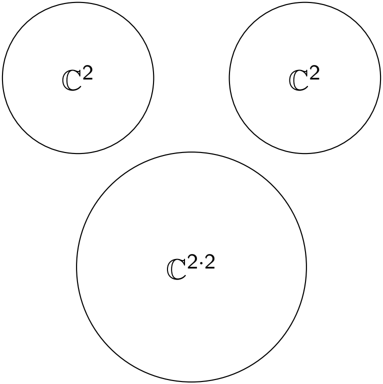
Quantum Computing 101: Composition



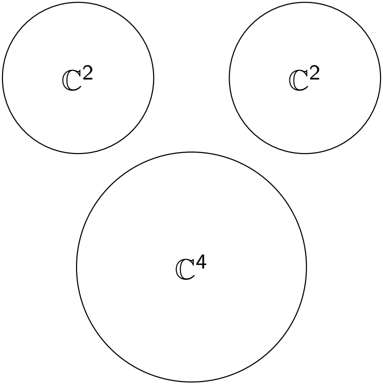
Quantum Computing 101: Composition



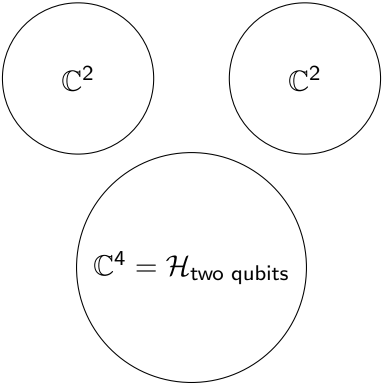
Quantum Computing 101: Composition



Quantum Computing 101: Composition



Quantum Computing 101: Composition



A Comparison

Quantum Computing 101: A Comparison

Table : Quantum mechanics and probability theory.

Probability Theory	Quantum Mechanics
Real numbers in $[0, 1]$	Complex numbers
Real numbers that sum to 1	Complex numbers that the squares of their magnitudes sum to 1
The *sum* is equal to 1	The *Euclidean norm* is equal to 1
The *sum* is preserved	The *Euclidean norm* is preserved
The L_1-norm is preserved	The L_2-norm is preserved
Use of stochastic matrices	Use of unitary matrices

Oracles

Quantum Computing 101: Oracles

classical oracle $f : \{0, 1\}^n \to \{0, 1\}$

quantum oracle $q_1 : |\psi\rangle \mapsto U |\psi\rangle$

CPTP quantum oracle $q_2 : \rho \mapsto \mathscr{U} \rho$

$$\rho = |\psi\rangle\langle\psi| \in \mathbb{C}^{2^n \times 2^n}$$

$$\mathscr{U} \rho = \sum_i A_i \rho A_i^\dagger$$

Quantum Computing 101: Oracles

classical oracle $f : \{0,1\}^n \rightarrow \{0,1\}$

Quantum Computing 101: Oracles

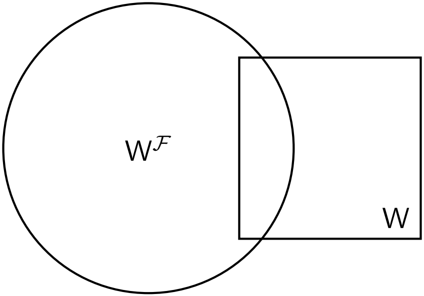


Figure : Our world, namely W, and a relativized world $W^{\mathcal{F}}$, induced by calls to some oracle family $\mathcal{F}$.

Quantum Complexity

Quantum Complexity

Some polynomial-time classes.

$$\begin{aligned} \mathsf{P} &\subseteq \mathsf{NP} \\ &\subseteq \mathsf{MA} \\ &\subseteq \mathsf{QMA} \\ &\subseteq \mathsf{PP} \end{aligned}$$

$$\begin{aligned} \mathsf{P} &\subseteq \mathsf{BPP} \\ &\subseteq \mathsf{BQP} \\ &\subseteq \mathsf{QMA} \end{aligned}$$

The Result

The Result

$$\exists \mathcal{F}\, \exists L : L \in \mathsf{NP}^{\mathcal{F}} \text{ and } L \notin \mathsf{BQP}^{\mathcal{F}}$$

The Result

$$\begin{aligned}
\mathcal{F} &= \{f_1, f_2, \ldots, f_n, \ldots\} \\
&= \{f_n \mid f_n : \{0,1\}^n \to \{0,1\}\} \\
&= \mathcal{F}_{\text{good}} \cup \mathcal{F}_{\text{bad}} \\
&= \mathcal{F}_{\text{good}} \uplus \mathcal{F}_{\text{bad}} \\
&= \{f_n \mid \exists x \in \{0,1\}^n : f_n(x) = 1\} \\
&\qquad\qquad \uplus \{f_n \mid \forall x \in \{0,1\}^n : f_n(x) = 0\}
\end{aligned}$$

$$\begin{aligned}
L &= \{1^n \mid f_n \text{ is good}\} \\
&= \{1^n \mid \exists x \in \{0,1\}^n : f_n(x) = 1\}
\end{aligned}$$

The Result

How do we use these Booleans f_n in the quantum world?

$$O_n : |x\rangle \mapsto (-1)^{f_n(x)} |x\rangle$$

The Result

We show $L \in \mathsf{NP}^{\mathcal{F}}$.

The Result: We show $L \in \mathsf{NP}^{\mathcal{F}}$

Given 1^n, we guess a $x \in \{0,1\}^n$.

If $f_n(x) = 1$, then $1^n \in L$, else $1^n \notin L$.

The Result

We show $L \notin \mathsf{BQP}^{\mathcal{F}}$.

The Result: We show $L \notin \mathrm{BQP}^{\mathcal{F}}$

$$f_n \in \mathcal{F} = \{f_n\}_n$$

$$\underline{f_n = h_n \text{ or } g_n?}$$

$$\forall y : h_n(y) = 0$$
$$\exists! x : g_n(x) = 1$$

$$h_n \in \mathcal{F}_{\text{bad}}$$
$$g_n \in \mathcal{F}_{\text{good}}$$

The Result: We show $L \notin \mathrm{BQP}^{\mathcal{F}}$

$$x, y \in \{0,1\}^n$$

$$O_{\mathbf{X}} : |y\rangle \mapsto (-1)^{h_n(y)} |y\rangle = |y\rangle$$

$$\forall y : h_n(y) = 0$$

$$O_{\mathbf{X}_x} : |y\rangle \mapsto (-1)^{g_n(y)} |y\rangle$$

$$\exists! x : g_n(x) = 1$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$x, y \in \{0,1\}^n$$

$$O_{\mathbf{X}} : |y\rangle \mapsto |y\rangle$$

$$O_{\mathbf{X}_x} : |y\rangle \mapsto |y\rangle$$
$$O_{\mathbf{X}_x} : |x\rangle \mapsto -|x\rangle$$

The Result: We show $L \notin \mathrm{BQP}^{\mathcal{F}}$

Suppose that we have two quantum computers with oracles $O_{\mathbf{X}}$, and $O_{\mathbf{X}_x}$.

Their initial states are $|\psi_0\rangle \in \mathcal{H}$ and $|\psi_0^x\rangle \in \mathcal{H}$, respectively.

We set

$$|\psi_0\rangle = |\psi_0^x\rangle .$$

The Result: We show $L \notin \mathrm{BQP}^{\mathcal{F}}$

$$|\psi_j\rangle = \sum_{y=1}^{2^n} \alpha_{y,j} |y\rangle$$

$$= \text{the state just before the } (j+1)\text{-th query of } O_{\mathbf{X}}.$$

$$O_{\mathbf{X}} : |y\rangle \mapsto |y\rangle$$

$$|\psi_j\rangle = U_j O_{\mathbf{X}} |\psi_{j-1}\rangle = U_j \mathbf{1} |\psi_{j-1}\rangle = U_j |\psi_{j-1}\rangle$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$\left|\psi_j^x\right\rangle = \sum_{y=1}^{2^n} \alpha_{y,j}^x \left|y\right\rangle$$

$$= \text{the state just before the } (j+1)\text{-th query of } O_{\mathbf{X}_x}.$$

$$O_{\mathbf{X}_x} : \left|y\right\rangle \mapsto \left|y\right\rangle$$
$$O_{\mathbf{X}_x} : \left|x\right\rangle \mapsto -\left|x\right\rangle$$

$$\left|\psi_j^x\right\rangle = U_j O_{\mathbf{X}_x} \left|\psi_{j-1}^x\right\rangle$$
$$\left|\tilde{\psi}_j^x\right\rangle = U_j \left|\psi_{j-1}^x\right\rangle$$

The Result: We show $L \notin \mathrm{BQP}^{\mathcal{F}}$

$$
\begin{aligned}
\left\| \left| \tilde{\psi}^x_{j+1} \right\rangle - \left| \psi^x_{j+1} \right\rangle \right\|_2 &= \left\| U_j \left| \psi^x_j \right\rangle - U_i O_{\mathbf{X}_x} \left| \psi^x_j \right\rangle \right\|_2 \\
&= \left\| U_j \left(\left| \psi^x_j \right\rangle - O_{\mathbf{X}_x} \left| \psi^x_j \right\rangle \right) \right\|_2 \\
&\leq \left| \det \left(U_j \right) \right| \left\| \left| \psi^x_j \right\rangle - O_{\mathbf{X}_x} \left| \psi^x_j \right\rangle \right\|_2 \\
&= \left\| \left| \psi^x_j \right\rangle - O_{\mathbf{X}_x} \left| \psi^x_j \right\rangle \right\|_2 \\
&= \left\| \sum_y \alpha^x_{y,j} \left| y \right\rangle - O_{\mathbf{X}_x} \sum_y \alpha^x_{y,j} \left| y \right\rangle \right\|_2 \\
&= \left\| \alpha^x_{x,j} \left| x \right\rangle - \left(-\alpha^x_{x,j} \left| x \right\rangle \right) \right\|_2 \\
&= \left\| 2\alpha^x_{x,j} \left| x \right\rangle \right\|_2 \\
&= \left| 2\alpha^x_{x,j} \right| \left\| \left| x \right\rangle \right\|_2 \\
&= 2 \left| \alpha^x_{x,j} \right|
\end{aligned}
$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$\left\| \left|\tilde{\psi}^{x}_{j+1}\right\rangle - \left|\psi^{x}_{j+1}\right\rangle \right\|_2 \leq 2 \left|\alpha^{x}_{x,j}\right|$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$\left\| |\psi_{j+1}\rangle - \left|\psi^{x}_{j+1}\right\rangle \right\| \le \, ?$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$
\begin{aligned}
\left\| |\psi_{j+1}\rangle - |\psi_{j+1}^x\rangle \right\| &\leq \left\| \left|\tilde{\psi}_{j+1}^x\right\rangle - |\psi_{j+1}\rangle \right\| + \left\| \left|\tilde{\psi}_{j+1}^x\right\rangle - |\psi_{j+1}^x\rangle \right\| \\
&\leq \left\| \left|\tilde{\psi}_{j+1}^x\right\rangle - |\psi_{j+1}\rangle \right\| + 2\left|\alpha_{x,j}^x\right| \\
&= \left\| U_j \left|\psi_j^x\right\rangle - U_j |\psi_j\rangle \right\| + 2\left|\alpha_{x,j}^x\right| \\
&= \left\| U_j \left(\left|\psi_j^x\right\rangle - |\psi_j\rangle\right) \right\| + 2\left|\alpha_{x,j}^x\right| \\
&\leq \left|\det\left(U_j\right)\right| \left\| \left|\psi_j^x\right\rangle - |\psi_j\rangle \right\| + 2\left|\alpha_{x,j}^x\right| \\
&= \left\| |\psi_j\rangle - \left|\psi_j^x\right\rangle \right\| + 2\left|\alpha_{x,j}^x\right|
\end{aligned}
$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$
\begin{aligned}
\left\| |\psi_{j+1}\rangle - \left|\psi_{j+1}^{x}\right\rangle \right\| &\leq \left\| |\psi_{j}\rangle - \left|\psi_{j}^{x}\right\rangle \right\| + 2\left|\alpha_{x,j}^{x}\right| \\
&\vdots \\
&\leq \underbrace{\left\| |\psi_{0}\rangle - |\psi_{0}^{x}\rangle \right\|}_{=0} + \sum_{k=0}^{j} 2\left|\alpha_{x,k}^{x}\right| \\
&= \sum_{k=0}^{j} 2\left|\alpha_{x,k}^{x}\right|
\end{aligned}
$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$\left\| |\psi_{j+1}\rangle - |\psi_{j+1}^x\rangle \right\| \leq \sum_{k=0}^{j} 2 \left| \alpha_{x,k}^x \right|$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$f_n \in \mathcal{F} = \{f_n\}_n$$

$$\underline{f_n = h_n \text{ or } g_n?}$$

$$\forall y : h_n(y) = 0$$
$$\exists! x : g_n(x) = 1$$

$$h_n \in \mathcal{F}_{\text{bad}}$$
$$g_n \in \mathcal{F}_{\text{good}}$$

The Result: We show $L \notin \mathrm{BQP}^{\mathcal{F}}$

T = the total number of oracle queries.

$$\Pr\left[\text{the answer is correct}\right] \geq \frac{2}{3} \Rightarrow \| |\psi_T\rangle - |\psi_T^x\rangle \| > \frac{1}{3}$$

Without proof!

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$\| |\psi_T\rangle - |\psi_T^x\rangle \| > \frac{1}{3}$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$\sum_{k=0}^{T-1} 2\left|\alpha_{x,k}^{x}\right| \geq \| |\psi_T\rangle - |\psi_T^x\rangle \| > \frac{1}{3}$$

The Result: We show $L \notin \mathrm{BQP}^{\mathcal{F}}$

$$\sum_{k=0}^{T-1} \left|\alpha_{x,k}^{x}\right| \geq \frac{1}{2} \left\| |\psi_T\rangle - |\psi_T^x\rangle \right\| > \frac{1}{6}$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$\sum_{k=0}^{T-1} \left|\alpha_{x,k}^{x}\right| > \frac{1}{6}$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$\sum_{k=0}^{T-1} \left|\alpha_{x,k}^{x}\right| > \frac{1}{6}$$

$$\sum_{x=1}^{N} \sum_{k=0}^{T-1} \left|\alpha_{x,k}^{x}\right| > \frac{N}{6} \tag{1}$$

$$N = 2^n$$

The Result: We show $L \notin \mathsf{BQP}^{\mathcal{F}}$

$$\underbrace{\left(\sum_{x=1}^{N} \alpha_{x,k}^{x} \left|x\right\rangle\right) \in \mathcal{H}}_{\text{By induction on } k \in \mathbb{N}.} \Leftrightarrow \sum_{x=1}^{N} \left|\alpha_{x,k}^{x}\right|^{2} = 1$$

The Result: We show $L \notin \mathrm{BQP}^{\mathcal{F}}$

$$\left(\sum_{x=1}^{N} \left|\alpha_{x,k}^{x}\right| \cdot 1\right)^2 \leq \left(\sum_{x=1}^{N} \left|\alpha_{x,k}^{x}\right|^2\right) \cdot \left(\sum_{x=1}^{N} 1^2\right)$$
$$= 1 \cdot N$$

$$\sum_{x=1}^{N} \left|\alpha_{x,k}^{x}\right| \leq \sqrt{N}$$
$$\sum_{k=0}^{T-1} \sum_{x=1}^{N} \left|\alpha_{x,k}^{x}\right| \leq T\sqrt{N}$$
$$\sum_{x=1}^{N} \sum_{k=0}^{T-1} \left|\alpha_{x,k}^{x}\right| \leq T\sqrt{N} \qquad (2)$$

The Result: We show $L \notin \mathrm{BQP}^{\mathcal{F}}$

$$\sum_{x=1}^{N} \sum_{k=0}^{T-1} \left|\alpha_{x,k}^{x}\right| > \frac{N}{6} \tag{1}$$

$$\sum_{x=1}^{N} \sum_{k=0}^{T-1} \left|\alpha_{x,k}^{x}\right| \leq T\sqrt{N} \tag{2}$$

$$\frac{N}{6} < T\sqrt{N}$$

$$T > \frac{\sqrt{N}}{6}$$

$$T \in \Omega\left(\sqrt{N}\right) = \Omega\left(2^{\frac{n}{2}}\right)$$

The Result: We show $L \notin \mathrm{BQP}^{\mathcal{F}}$

$$f_n \in \mathcal{F} = \{f_n\}_n$$

$$\underline{f_n = h_n \text{ or } g_n?}$$

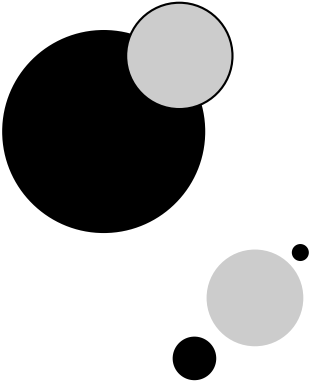
$$\forall y : h_n(y) = 0$$
$$\exists! x : g_n(x) = 1$$

$$h_n \in \mathcal{F}_{\text{bad}}$$
$$g_n \in \mathcal{F}_{\text{good}}$$

Conclusion

Conclusion: What have we learned?

Thank You!



Appendix

Appendix: Random Bits

$$|0\rangle \; -\boxed{H}- \quad \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\begin{aligned}
\Pr[\text{the outcome is } 0] &= \left|\frac{1}{\sqrt{2}}\right|^2 \\
&= \frac{1}{2} \\
&= \left|\frac{1}{\sqrt{2}}\right|^2 \\
&= \Pr[\text{the outcome is } 1]
\end{aligned}$$